Boundedness in general type MMP

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2025 年 7 月 26 日

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Conjecture: Termination of the Minimal Model Program

For any smooth projective variety X **after finitely many steps of MMP** (divisorial contractions and flips), we reach either a "minimal model" Y (with terminal singularities) of X: K_Y is nef ($K_Y \cdot C \geq 0$ for any curve C) or a Mori fiber space: Y admits a Fano fibration $Y \to Z$ ($-K_Y$ is ample over Z, and $\dim Z < \dim Y$),

$$X_0 := X \dashrightarrow X_1 \dashrightarrow \cdots \dashrightarrow X_n := Y.$$

- Usual MMP is for pairs (X, B), for simplicity, this talk will focus on $B = 0$.

- Termination of MMP for surfaces is easy as the Picard number ρ drops after each divisorial contraction (no flip for surfaces).
- Termination of MMP in dimension 3 by
  - Shokurov 85' (X is terminal), introduced so called "difficulty function", the function drops after each step of flip:

    $$d(X) := \#\{E \mid A_X(E) < 2\} < +\infty,$$

    let $X \dashrightarrow X^+$ be a flip, and E the exceptional divisor of the blow up the flipped curve C^+, $A_{X^+}(E) = 2 > A_X(E)$, and $d(X) > d(X^+)$.
  - Kawamata 92' (X is klt, "difficulty function")
  - Shokurov 96' (X is lc, "special termination", slogan: termination in low dimension implies termiantion near lc locus).

When $d = \dim X \geq 4$, "difficulty function" does not work well:

- $(d-m, d-n)$-flips may appear instead of $(d-2, d-2)$-flip, $n > 2$, where $d-n$ is the dimension of the flipped locus.
- Blow-up flipped locus: $A_{X^+}(E) = n > 2$ but $\#\{E \mid A_X(E) < n\}$ may not be finite.

When $\dim X = 4$, termination is known when

- X is terminal (Kawamata-Matsuda-Matsuki 88', Fujino 05'): "difficulty function" works for $(1,2)$, $(2,2)$-flip. For $(2,1)$-flip, $h_4^{\mathrm{alg}}(X)$ drops.
- $-K_X \equiv D \geq 0$ for some D (Alexeev-Hacon-Kawamata 07'): new "(weighted) difficulty function".

When $\dim X = 5$, $(2,2)$-flip for terminal X is unknown.

When $\dim X \geq 4$, and $K_X \equiv D \geq 0$, Birkar 07': termination in low dimension implies the termination of K_X-MMP.

Birkar's idea:

- Any K_X-MMP is a $(K_X + tD)$-MMP for any $t \geq 0$, where
- $t_X = \mathrm{lct}(X; D) \leq \mathrm{lct}(X^+; D^+) = t_{X^+}$, where $\mathrm{lct}(X; D) := \max\{t \geq 0 \mid (X, tD) \text{ is lc.}\}$,
- (assume termination in low dimension) Special termination: termination near lc locus of $(X, t_X D)$, outside lc locus $\mathrm{lct}(X; D)$ increases.
- Repeat special termination, get a strictly increasing sequence of $\mathrm{lct}(X; D)$ which contradicts the ACC for lct's.

H.-J. Liu-Qi-Zhuang 2025 preprint

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- Shokurov's "difficulty function" does not work well in dimension ≥ 5.
- Termination in dimension 4 is unknown in full generality.

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- Termination in dimension 4 is unknown in full generality.
- Idea: apply idea of [Birkar 07'], termination of terminal fourfold, and tools from local (algebraic) K-stability theory.

For any birational morphism $f : Y \to X$, we may write

$$K_Y + \sum_E (1 - A_X(E))E \sim_{\mathbb{Q}} f^* K_X,$$

where E run over exceptional divisors of f.

Termination: Shokurov's approach-minimal log discrepancies

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Let x be a closed point on X, the *minimal log discrepancy* (mld) is defined by:

$$\mathrm{mld}(X \ni x) := \min\{A_X(E) \mid \forall E \, \mathrm{center}_X(E) = \{x\}\}.$$

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- Mld measures the singularities: $\mathrm{mld}(X \ni x)$ is larger if singularity $x \in X$ is better.
- Conjecture: $\mathrm{mld}(X \ni x) \leq \dim X$ with the equality holds iff X is smooth near x.
- $\dim X = 2$, the set of mlds contains $\{\frac{1}{n} \mid n \in \mathbb{N}_+\}$, and if $\mathrm{mld}(X \ni x) < 1$, then $\mathrm{mld}(X \ni x) \leq \frac{2}{3}$.

Conjecture (ACC conjecture for mlds, Shokurov 1988)

For any $x \in X$ of a given dimension d, $\mathrm{mld}(X \ni x)$ belongs to a set which satisfies the ascending chain condition (ACC).

Conjecture (LSC conjecture for mlds, Ambro 1999)

Let X be a variety with mild (klt) singularities. Then the function $x \mapsto \mathrm{mld}(X \ni x)$ is lower-semicontinuous (LSC).

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- Reduce a global problem (classification of varieties) to a local problem (singularities).

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• **Q**: Are ACC for mlds and LSC for mlds harder than termination? Is there any other approach?

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[H.-J. Liu 2025] show the following:
• ACC for mlds for exceptionally non-canonical (enc) singularities and both conjectures for terminal singularities are enough.
• enc: non-canonical, and exactly one non-terminal exceptional divisor.

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- enc: non-canonical, and exactly one non-terminal exceptional divisor.
- ACC for enc singularities holds in dimension 3, so termination holds for threefolds (without difficult function).
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- In order to prove the termination for fourfolds, it suffices to show ACC for mlds for enc singularities.
- enc singularities in dimension 4 seems still be too complicated.

Termination: any other approach?

I asked the following question in 2017 when I was a Ph.D. student (in my Research Statement when I applied for postdoc).

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- C. Li introduced local volumes with motivation from questions in K-stability.
- LSC holds for local volumes (Blum-Y. Liu 2021).
- ACC holds for local volumes (Xu-Zhuang 2024 preprint, H.-J. Liu-Qi 2024 preprint)

Let X be a klt variety of dimensional n and $x \in X$ a closed point. For any $v \in \mathrm{Val}_{X,x}$, the *local volume* of X, at x is defined as

$$\widehat{\mathrm{vol}}(x, X) := \inf_{v \in \mathrm{Val}_{X,x}} \widehat{\mathrm{vol}}_X(v) > 0,$$

where $\widehat{\mathrm{vol}}_X(v)$ is the *normalized volume* of v:

$$\widehat{\mathrm{vol}}_X(v) := \begin{cases} A_X(v)^n \cdot \mathrm{vol}_{X,x}(v), & \text{if } A_X(v) < +\infty, \\ +\infty, & \text{if } A_X(v) = +\infty. \end{cases}$$

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- Key differences: mlds satisfy monotonicity in the MMP, while the local volume does not:
- It is possible for a flip $(X, \Delta) \dashrightarrow (X^+, \Delta^+)$: $\widehat{\mathrm{vol}}_{(X,\Delta)}(v) > \widehat{\mathrm{vol}}_{(X^+,\Delta^+)}(v)$.

Solution: introduce *log canonical local volumes*: $\widehat{\mathrm{Vol}}_X$.

Key Lemmas, H.-J. Liu-Qi-Zhuang 2025 preprint

- $\varphi\colon X \dashrightarrow X'$ be an MMP type contraction between general type (K_X is big) klt varieties. Then $\widehat{\mathrm{Vol}}_{X'} \geq \widehat{\mathrm{Vol}}_X$.
- Let X be a general type klt variety. Then for any closed point $x \in X$, $\widehat{\mathrm{vol}}(x, X) \geq \widehat{\mathrm{Vol}}_X$.

As a consequence:

H.-J. Liu-Qi-Zhuang 2025 preprint, Theorem A

Let X be a general type projective klt variety. Then there exists $\varepsilon > 0$ such that for any sequence of steps of a K_X-MMP $X \dashrightarrow X'$ and any closed point $x' \in X'$, $\widehat{\mathrm{vol}}(x', X') \geq \varepsilon$.

Xu-Zhuang 21

Let D be a $\mathbb{Q}$-Cartier Weil divisor on X. Then rD is Cartier near x for some positive integer $r \leq \frac{n^n}{\widehat{\mathrm{vol}}(x,X)}$.

Combing Xu-Zhuang and Theorem A, we obtain:

H.-J. Liu-Qi-Zhuang 2025 preprint, Theorem B

Let X be a general type (K_X is big) projective klt variety. There exist a positive integer r and a finite set $S \subseteq \mathbb{R}_+$, depending only on X, such that for any sequence of steps of a K_X-MMP $X \dashrightarrow Y$,

1. the Cartier index of any $\mathbb{Q}$-Cartier Weil divisor on Y is at most r, and
2. for any point $y \in Y$, $\mathrm{mld}(y, Y) \in S$.

• Theorem B confirms the ACC conjecture of mld for general type MMPs.

H.-J. Liu-Qi-Zhuang 2025 preprint, Theorem C

Let X be a general type projective klt variety of dimension 5. There exists a positive integer m depending only on X such that every K_X-MMP terminates after at most m steps.

Idea of the proof:

- Apply idea of [Birkar 07] and Theorem B, it suffices to prove the termination in dimension 4 and the Cartier index of K_X are bounded in the MMP.
- Lift the MMP to the terminalization, we may reduce to the terminal fourfolds case.

Although we are unable to prove the termination of general type MMP, we show the boundedness:

H.-J. Liu-Qi-Zhuang 2025 preprint, Theorem D

Let X be a general type projective klt variety. Then there exists a projective family $\mathcal{W} \to \mathcal{B}$ over a finite type base $\mathcal{B}$, such that in any sequence of K_X-MMP, every fiber of the extremal contractions or the flips is isomorphic to $\mathcal{W}_b$ for some $b \in \mathcal{B}$.

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• Our method can be used to show that any fixed order infinitesimal neighborhood of the fibers belong to a bounded family.
• We expect: there are only finitely many "analytic types" of flips in the MMP, and the invariants of local nature can only change in finitely many different ways in the MMP.

H.-J. Liu-Qi-Zhuang 2025 preprint, Theorem E

Let N be a positive integer and X a general type projective smooth variety of dimension 4. Assume that

1. The Picard number of X is at most N, $h_4^{\mathrm{alg}}(X) \leq N$, and
2. $\mathrm{vol}(K_X) \geq \frac{1}{N}$, and $(K_X \cdot H^3) \leq N$ for some very ample divisor H on X.

Then any K_X-MMP terminates after at most M^M steps, where $M := (2N)^9!$.

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• Here $h_4^{\mathrm{alg}}(X)$ is the dimension of the subgroup in the singular homology group $H_4(X, \mathbb{R})$ generated by algebraic cycles.

• We also obtain explicit termination result for threefolds (not necessarily general type).

• We prove Theorems A-E for klt pairs (with big boundary Δ), and even the termination was unknown for such pairs (in dimension 4).

Thank you!!

- A graded linear series $V_\bullet$ is *eventually birational* if the induced rational map $X \dashrightarrow |V_m|$ is birational onto its image for all sufficiently large $m \in M(V_\bullet)$.
- Let L be a big $\mathbb{R}$-Cartier divisor. A graded linear series $V_\bullet$ of L is called *admissible* if it is eventually birational and (X, Γ) is log canonical for all $m \in M(V_\bullet)$ and all $\Gamma \in \frac{1}{m}|V_m|$.
- Define the *log canonical volume* of L as follows:

$$\widehat{\mathrm{Vol}}_X(L) := \sup\{\mathrm{vol}(V_\bullet) \mid V_\bullet \text{ is admissible}\},$$

where $\mathrm{vol}(V_\bullet) := \lim_{M(V_\bullet)\ni m\to+\infty} \frac{\dim(V_m)}{m^n/n!}$.